

Problem Set 7

Discrete Structures

Due on the 17th day of April of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order*; solutions to earlier problems may be applied as theorems in the proofs of later problems, but *not vice versa*. In this problem set, whenever you *define a function*, you are **not** required to *prove* that the thing you've defined is in fact a function (as long as you *have* actually defined a function). You may also rely on the *Cantor-Schröder-Bernstein* theorem as stated below.

Theorem: Cantor-Schröder-Bernstein.

$$\forall x \forall y ((|x| \leq |y| \wedge |y| \leq |x|) \Rightarrow |x| = |y|).$$

We define a set X to be **finite** iff $(\exists n \in \mathbb{N})(|X| = |n|)$, meaning that X can be put in bijection with the set $\{0, 1, \dots, n-1\}$ for some $n \in \mathbb{N}$. In such a situation, we will define the **cardinality of X** to be $|X| := n$, so saying $|X| \in \mathbb{N}$ is equivalent to saying “ X is finite.”

Given this definition, you may also rely on the following theorems in this problem set.

Lemma.

If X is a finite set and Y is a set such that $Y \subseteq X$, then Y is also finite.

Theorem.

For any finite sets X and Y , if $Y \subseteq X$, then $|X \setminus Y| = |X| - |Y|$.

Corollary.

For any nonempty finite set X , and for any $x \in X$, we have $|X \setminus \{x\}| = |X| - 1$.

Given sets A, B, C and functions $f : A \rightarrow B$ and $g : B \rightarrow C$, we define the **composition** of f with g to be the function $g \circ f : A \rightarrow C$ given by $(g \circ f)(a) := g(f(a))$ for all $a \in A$.

1. a. Show that $\forall X \forall Y (X \subseteq Y \Rightarrow |X| \leq |Y|)$.
 b. Let X and Y be sets such that $X \neq \emptyset$ and $Y \neq \emptyset$, and let $f : X \rightarrow Y$ be an injection. Prove that there exists a function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$.¹
2. a. Prove that $\forall A \forall B \forall C ((|A| \leq |B| \wedge |B| \leq |C|) \Rightarrow |A| \leq |C|)$.
 b. Prove that $\forall A \forall B \forall C ((|A| \geq |B| \wedge |B| \geq |C|) \Rightarrow |A| \geq |C|)$.
 c. Prove that $\forall A \forall B \forall C ((|A| = |B| \wedge |B| = |C|) \Rightarrow |A| = |C|)$.
3. For any finite sets A and B , prove that $|A \cup B| = |A| + |B|$ when $A \cap B = \emptyset$.
4. For any finite set X , show that $|\mathbb{P}(X)| = 2^{|X|}$.²

¹This problem is asking you to prove that every injective function has a left inverse. The dual statement—every surjective function has a right inverse—is equivalent to the *axiom of choice*.

²Hint: try induction.