Monotone Catenary Degree of Numerical Monoids

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Motivation

- In general, not much is known about the monotone catenary degree of numerical monoids. In the past, the monotone catenary degree in Krull Monoids has been studied, but for numerical monoids, only the regular catenary degree is well understood.
- We seek to gain a deeper understanding of the equivalent and adjacent catenary degrees in order to characterize the relationship between monotone and regular catenary degrees of numerical monoids.
- It is known that, for a numerical monoid M, $c_{mon}(M) \geq c(M)$. We aim to determine when this inequality is strict, and when the two quantities are equal.

Numerical Monoids

- A **Numerical Monoid** is a cofinite subset of the nonnegative integers closed under the operation of addition. It is known that for every numerical monoid M, there exists a minimal set of generators $n_1, \ldots, n_k$, so for a monoid of this form we will write

$$M = \langle n_1, \ldots, n_k \rangle = \{a_1 n_1 + \ldots + a_k n_k \mid (a_1, \ldots a_k) \in \mathbb{N}^k\}$$

Example

$M = \langle 4, 9, 11 \rangle = \{0, 4, 8, 9, 11, 12, 13, 15, \ldots\}$

- The **Set of Factorizations** of an element $m \in M$ is defined as

$$\mathcal{Z}(m) = \{(a_1, \ldots, a_k) \in \mathbb{N}^k \mid a_1 n_1 + \ldots + a_k n_k = m\}$$

Example

For $M = \langle 4, 9, 11 \rangle$, $\mathcal{Z}(26) = \{(2, 2, 0), (0, 1, 2)\}$

- The **Length** of a factorization $z = (z_1, \ldots, z_k) \in \mathcal{Z}(m)$ is defined as

$$|z| = z_1 + \cdots + z_k$$

Example

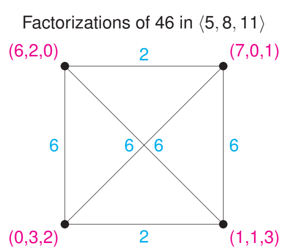
Consider $(2, 2, 0) \in \mathcal{Z}(26)$ $|2, 2, 0| = 4$

- We can define a **Distance** between factorizations based on the differences of the coordinates of the factorizations.

Example

$d((2, 2, 0), (0, 1, 2)) = 3$

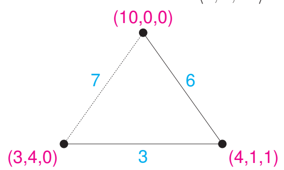
Factorization Graph



Catenary Degrees

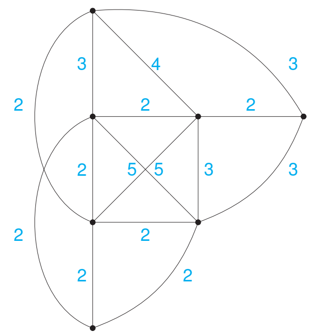
- Two factorizations z and z' are connected by an **N-chain** if there exists a sequence $z_0, \ldots, z_k \in \mathcal{Z}(m)$ such that $z_0 = z, \ldots, z_k = z'$ and $d(z_i, z_{i+1}) \leq N$ for all $i \in \{1, \ldots k-1\}$.
- The **catenary degree** of an element $m \in M$, $c(m)$, is the minimum natural number N such that there is an N-chain between any two factorizations of m.
- Besides the regular catenary degree, we are also concerned with the equivalent, adjacent, and monotone catenary degrees. These behave similarly to the regular catenary degree but with added restrictions.

Catenary Graph



- We can remove the edge of length 7 and our graph stays connected.
- If we remove the edge of length 6, our graph will be disconnected
- $c(40) = 6$

Different Types of Catenary Degrees

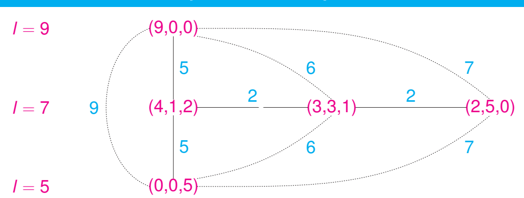


- Equivalent Catenary Degree only deals with factorizations of the same length
- Adjacent Catenary Degree deals with moving from one length factorization to the next
- Monotone Catenary Degree is the maximum of these two values

When is $c_{mon}(M) = c(M)$?

For any monoid generated by an arithmetic sequence, $M = \langle a, a+d, \ldots, a+kd \rangle$, $c_{mon}(M) = c(M)$.
Furthermore, for any element $m \in M$, $c_{mon}(m) = c(m)$.

Monoids Generated by Arithmetic Sequences



Do Generalized Arithmetic Monoids Behave the Same Way?

In monoids generated by generalized arithmetic sequences $M = \langle a,\ ah + d,\ ah + 2d \rangle$ the monotone catenary is more nuanced.
Our research has led us to make the following observations:

- If $\gcd(h-1, d) > 1$, then we have that $c(M) = c_{mon}(M)$.
- if $\gcd(h-1, d) = 1$, we have several cases:
  - If $h < d$, then $c(M) < c_{mon}(M)$.
  - If $h \geq d$ and $c(M) < c_{eq}(M)$, then $c(M) < c_{mon}(M)$
  - If $h \geq d$ and $c(M) = c_{eq}(M)$, then $c(M) = c_{mon}(M)$

When is $c_{mon}(M) > c(M)$?

Given that the following two conditions hold, $c_{mon}(s) > c(s)$ for any element s in any monoid M.

- $c_{eq}(s) > c_{adj}(s)$
- If s has two factorizations z_1 and z_2 of length l, there exists a factorization z_3 of length $q \neq l$ such that $d(z_1, z_3) < c_{eq}(s)$ and $d(z_2, z_3) < c_{eq}(s)$.

Case Where $c_{mon}(M) > c(M)$

Let $M = \langle na, na + n, 2na + nx + 1 \rangle$ with $x \geq 2$. Then

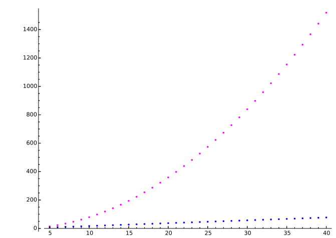
- $c_{eq}(M) = na + nx + 1$
- $c_{adj}(M) < na + nx + 1$
- $c_{mon}(M) = na + nx + 1$
- $c_{mon}(M) > c(M)$

Example

Let $M = \langle 8, 10, 21 \rangle$, so $n = 2$, $a = 4$, and $x = 2$. Then $c_{eq}(M) = c_{mon}(M) = 13$, and $c(M) = 5$. Then $c_{mon}(M) > c(M)$.

$c_{mon}(M) - c(M)$ can be rbitrarily Large

Monotone and Regular Catenary Degrees of $\langle a, a+1, \mathcal{F}\langle a, a+1 \rangle \rangle$



- $c_{mon}(M) = a^2 - 2a - 1$
- $c(M) = 2a - 3$
- $c_{mon}(M) - c(M) = a^2 - 4a - 4$

For large a, this difference can grow arbitrarily large.

Conclusions

- In some monoids, namely those generated by arithmetic sequences, $c_{mon}(M) = c(M)$.
- In generalized arithmetic monoids, we can have either $c_{mon}(M) > c(M)$ or $c_{mon}(M) = c(M)$.
- In general, we expect that $c_{mon}(M) > c(M)$. In fact, the difference between the two can grow arbitrarily large.

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